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About O Level

ADDITIONAL MATHEMATICS
(TOPICAL)

About Thinking Process

In solving mathematical problems, we always work backward. After indentifying our main target, we go 'backward' to look for the 'easier' targets until we are able to solve the problems.

Thinking process reveals how the teacher actually goes about solving a sum in the above-said manner.

About Teacher's Comments

It reveals the extra but relevant information which is not required as part of the solutions but are extremely useful in knowing how the solutions are arrived.

period	**2006 to 2022**
contents	**June & November, Paper 1 & 2, Worked Solutions**
form	**Topic By Topic**
compiled for	**O Levels**
special features	**Thinking Process, Teacher's Comments**

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Topic 6

Logarithmic & Exponential Functions

1 *(J06/P1/Q9)*

Given that $u = \log_4 x$, find, in simplest form in terms of u,

(i) x,

(ii) $\log_4\left(\dfrac{16}{x}\right)$,

(iii) $\log_x 8$. [5]

Thinking Process

(i) Change the given logarithmic expression into index form.

(ii) Apply logarithmic rule: $\lg\left(\dfrac{a}{b}\right) = \lg a - \lg b$.

(iii) Change base for $\log_x 8$.

Solution

(i) $u = \log_4 x$

$\Rightarrow x = 4^u$

$\Rightarrow x = 2^{2u}$

(ii) $\log_4\left(\dfrac{16}{x}\right) = \log_4 16 - \log_4 x$

$= \log_4 4^2 - \log_4 x = 2 - u$

(iii) $\log_x 8 = \dfrac{\log_4 8}{\log_4 x}$

$= \dfrac{1}{\log_4 x} \times (\log_4 2^3)$

$= \dfrac{3}{\log_4 x} \times (\log_4 2)$

$= \dfrac{3}{u}\left(\log_4 4^{\frac{1}{2}}\right) = \dfrac{3}{u}\left(\dfrac{1}{2}\right) = \dfrac{3}{2u}$

2 *(D06/P1/Q8)*

(a) Solve the equation $\lg(x+12) = 1 + \lg(2-x)$. [3]

(b) Given that $\log_2 p = a$, $\log_8 q = b$ and $\dfrac{p}{q} = 2^c$, express c in terms of a and b. [4]

Thinking Process

(a) Apply "Quotient law of Logarithms" and write 1 as lg 10.

(b) Express p and q as base of 2.

Solution

(a) $\lg(x+12) = 1 + \lg(2-x)$

$\Rightarrow \lg(x+12) - \lg(2-x) = 1$

$\Rightarrow \lg\dfrac{x+12}{2-x} = \lg 10$

$\Rightarrow \dfrac{x+12}{2-x} = 10$

$\Rightarrow x + 12 = 20 - 10x$

$\Rightarrow 11x = 8$

$x = \dfrac{8}{11}$

(b) $\log_2 p = a$ $\qquad$ $\log_8 q = b$

$\Rightarrow p = 2^a$ $\qquad$ $\Rightarrow q = 8^b$

$\qquad\qquad\qquad$ $\Rightarrow q = 2^{3b}$

$\dfrac{p}{q} = 2^a \div 2^{3b}$

$\Rightarrow 2^c = 2^{a-3b}$

$\therefore c = a - 3b$

3 *(N07/P1/Q7)*

(i) Use the substitution $u = 2^x$ to solve the equation $2^{2x} = 2^{x+2} + 5$. [5]

(ii) Solve the equation

$$2\log_9 3 + \log_5(7y-3) = \log_2 8.$$ [4]

Thinking Process

(i) Substitute 2^x as u and solve for u. Using log subsequently, solve for x.

(ii) Apply log $m^n = n\log m$ and $\log_a a = 1$.

Solution

(i) $2^{2x} = 2^{x+2} + 5 \Rightarrow (2^x)^2 = 2^x \times 2^2 + 5$

substituting $u = 2^x$, we have

$(u)^2 = u \times 2^2 + 5$

$u^2 - 4u - 5 = 0$

$(u-5)(u+1) = 0$

$\Rightarrow u - 5 = 0$ or $u + 1 = 0$

$\Rightarrow u = 5$ or $u = -1$

$\Rightarrow 2^x = 5$ $\qquad$ $2^x = -1$ (rejected)

taking lg both sides

$\lg 2^x = \lg 5$

$x \lg 2 = \lg 5$

$x = \dfrac{\lg 5}{\lg 2} = 2.32$ (3 sf) **Ans.**

(ii) $2\log_9 3 + \log_5(7y-3) = \log_2 8$

$$\log_9 3^2 + \log_5(7y-3) = \log_2 2^3$$

$$\log_9 9 + \log_5(7y-3) = 3\log_2 2$$

$$1 + \log_5(7y-3) = 3$$

$$\log_5(7y-3) = 2$$

$$7y - 3 = 5^2$$

$$7y - 3 = 25$$

$$7y = 28 \Rightarrow y = 4 \quad \textbf{Ans.}$$

4 *(J08/P1/Q8)*

(i) Given that $\log_9 x = a\log_3 x$, find a. [1]

(ii) Given that $\log_{27} y = b\log_3 y$, find b. [1]

(iii) Hence solve, for x and y, the simultaneous equations

$$6\log_9 x + 3\log_{27} y = 8,$$

$$\log_3 x + 2\log_9 y = 2.$$ [4]

Thinking Process

(i) & (ii) Apply change of base i.e. $\log_b a = \dfrac{\log_c a}{\log_c b}$

(iii) ✐ Use results of part (i) & (ii) to solve (iii).

Solution

(i) $\log_9 x = a\log_3 x$

$$a = \log_9 x \times \frac{1}{\log_3 x}$$

$$= \frac{\log_3 x}{\log_3 9} \times \frac{1}{\log_3 x}$$

$$= \frac{\log_3 x}{2\log_3 3} \times \frac{1}{\log_3 x} = \frac{1}{2}$$

$\therefore \quad a = \frac{1}{2}$ **Ans.**

(ii) $\log_{27} y = b\log_3 y$

$$b = \log_{27} y \times \frac{1}{\log_3 y}$$

$$b = \frac{\log_3 y}{\log_3 27} \times \frac{1}{\log_3 y}$$

$$b = \frac{\log_3 y}{3\log_3 3} \times \frac{1}{\log_3 y} = \frac{1}{3}$$

$\therefore \quad b = \frac{1}{3}$ **Ans.**

(iii) Consider $6\log_9 x + 3\log_{27} y = 8$

using results from part (i) and (ii)

$$6(a\log_3 x) + 3(b\log_3 y) = 8$$

$$\Rightarrow \quad 6\left(\frac{1}{2}\log_3 x\right) + 3\left(\frac{1}{3}\log_3 y\right) = 8$$

$$\Rightarrow \quad 3\log_3 x + \log_3 y = 8 \cdots\cdots\cdots(1)$$

consider $\log_3 x + 2\log_9 y = 2$

using results from part (i)

$$\log_3 x + 2\left(\frac{1}{2}\log_3 y\right) = 2$$

$$\log_3 x + \log_3 y = 2 \cdots\cdots\cdots(2)$$

from eq. (2), $\log_3 x = 2 - \log_3 y$, put in eq. (1)

$$3(2 - \log_3 y) + \log_3 y = 8$$

$$6 - 3\log_3 y + \log_3 y = 8$$

$$6 - 2\log_3 y = 8$$

$$2\log_3 y = -2$$

$$\log_3 y = -1$$

writing in index form

$$y = 3^{-1} = \frac{1}{3}$$

putting the value of y in eq. (2)

$$\log_3 x + \log_3(\tfrac{1}{3}) = 2$$

$$\log_3 x + \log_3(3^{-1}) = 2$$

$$\log_3 x - \log_3 3 = 2$$

$$\log_3 x - 1 = 2$$

$$\log_3 x = 3$$

writing in index form, $x = 3^3 = 27$

$\therefore \quad x = 27, \; y = \frac{1}{3}$ **Ans.**

5 *(J08/P2/Q8)*

Solve the equation

(i) $2^{2x+1} = 20$, [3]

(ii) $\dfrac{5^{4y-1}}{25^y} = \dfrac{125^{y+3}}{25^{2-y}}$. [4]

Thinking Process

(i) Express the equation in the form $A^x = B$. Take lg on both sides and solve for x.

(ii) Express 25 as 5^2, and 125 as 5^3. Solve for y.

Solution

(i) $2^{2x+1} = 20$

Taking log on both sides.

$$\lg 2^{2x+1} = \lg 20$$
$$(2x+1)\lg 2 = \lg 20$$
$$2x+1 = \frac{\lg 20}{\lg 2}$$
$$2x = 4.3219 - 1$$
$$x = \frac{3.3219}{2}$$
$x = 1.66$ (3sf) **Ans.**

(ii) $$\frac{5^{4y-1}}{25^y} = \frac{125^{y+3}}{25^{2-y}}$$
$$\frac{5^{4y-1}}{(5^2)^y} = \frac{(5^3)^{y+3}}{(5^2)^{2-y}}$$
$$\frac{5^{4y-1}}{5^{2y}} = \frac{5^{3y+9}}{5^{4-2y}}$$
$$5^{4y-1-2y} = 5^{3y+9-4+2y}$$
$$5^{2y-1} = 5^{5y+5}$$
$\Rightarrow \quad 2y - 1 = 5y + 5$
$$3y = -6$$
$y = -2$ **Ans.**

6 *(N08/P1/Q6)*

(i) Solve the equation $2t = 9 + \frac{5}{t}$. [3]

(ii) Hence, or otherwise, solve the equation

$2x^{\frac{1}{2}} = 9 + 5x^{-\frac{1}{2}}$. [3]

Thinking Process

(ii) To find x ✍ substitute $x^{\frac{1}{2}} = t$, form a quadratic equation and solve.

Solution

(i) $$2t = 9 + \frac{5}{t}$$
$$2t = \frac{9t+5}{t}$$
$$2t^2 = 9t + 5$$
$$2t^2 - 9t - 5 = 0$$
$$2t^2 - 10t + t - 5 = 0$$
$$2t(t-5) + 1(t-5) = 0$$
$$(t-5)(2t+1) = 0$$
$\Rightarrow \quad t - 5 = 0 \quad$ or $\quad 2t + 1 = 0$

$\therefore \quad t = 5 \quad$ or $\quad t = -\frac{1}{2}$ **Ans.**

(ii) $2x^{\frac{1}{2}} = 9 + 5x^{-\frac{1}{2}} \quad \Rightarrow \quad 2x^{\frac{1}{2}} = 9 + \frac{5}{x^{\frac{1}{2}}}$

Let $x^{\frac{1}{2}} = t$

$\Rightarrow \quad 2t = 9 + \frac{5}{t}$
$$2t^2 - 9t - 5 = 0$$
$$(t-5)(2t+1) = 0$$
$\therefore \quad t = 5, \quad$ or $\quad t = -\frac{1}{2}$

$\Rightarrow \quad x^{\frac{1}{2}} = 5 \quad$ or $\quad x^{\frac{1}{2}} = -\frac{1}{2}$

$\Rightarrow \quad x = 25 \quad$ or $\quad x = \frac{1}{4}$ **Ans.**

7 *(N08/P2/Q5)*

Solve the equation

(i) $\frac{4^x}{2^{5-x}} = \frac{2^{4x}}{8^{x-3}}$, [3]

(ii) $\lg(2y + 10) + \lg y = 2$. [3]

Thinking Process

(i) To find value of x ✍ Express the equation as a base of 2. Note that if $x^m = x^n \Rightarrow m = n$.

(ii) Recall: $\lg a + \lg b = \lg ab$, $x \lg a = \lg a^x$

Solution *with* TEACHER's COMMENTs

(i) $$\frac{4^x}{2^{5-x}} = \frac{2^{4x}}{8^{x-3}}$$
$$\frac{(2^2)^x}{2^{5-x}} = \frac{2^{4x}}{(2^3)^{x-3}}$$
$$2^{2x} \times 2^{-5+x} = 2^{4x} \times 2^{-3x+9}$$
$$2^{2x-5+x} = 2^{4x-3x+9}$$
$$2^{3x-5} = 2^{x+9}$$
$\Rightarrow \quad 3x - 5 = x + 9$
$$2x = 14$$
$x = 7$ **Ans.**

> $(a^m)^n = a^{mn}$
> $\frac{1}{a^x} = a^{-x}$
> $a^m \times a^{-n} = a^{m-n}$

(ii) $\lg(2y+10) + \lg y = 2$

$\lg[(2y+10)y] = 2\lg 10$ $\quad$ ($\because \lg 10 = 1$)

$\lg(2y^2 + 10y) = \lg 10^2$

$2y^2 + 10y = 100$

$y^2 + 5y - 50 = 0$

$y^2 + 10y - 5y - 50 = 0$

$y(y+10) - 5(y+10) = 0$

$(y+10)(y-5) = 0$

$y = 5, \quad y = -10$ (rejected)

$\therefore \quad y = 5$ **Ans.**

> Note that log of negative values are not defined.

9 *(J09/P2/Q7)*

Given that $\log_p X = 9$ and $\log_p Y = 6$, find

(i) $\log_p \sqrt{X}$, [1]

(ii) $\log_p\left(\frac{1}{X}\right)$, [1]

(iii) $\log_p(XY)$, [2]

(iv) $\log_Y X$. [2]

Thinking Process

(i) ✎ $\log_a m^n = n\log_a m$

(ii) ✎ $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) ✎ $\log_a(mn) = \log_a m + \log_a n$

(iv) ✎ Change of base rule.

Solution

(i) $\log_p \sqrt{X} = \frac{1}{2}\log_p X$

$= \frac{1}{2}(9) = 4.5$ **Ans.**

(ii) $\log_p\left(\frac{1}{X}\right) = \log_p 1 - \log_p X$

$= 0 - 9 = -9$ **Ans.**

(iii) $\log_p(XY) = \log_p X + \log_p Y$

$= 9 + 6 = 15$ **Ans.**

(iv) $\log_Y X = \dfrac{\log_p X}{\log_p Y}$

$= \frac{9}{6} = 1.5$ **Ans.**

10 *(N09/P2/Q10 a)*

Solve $\lg(7x-3) + 2\lg 5 = 2 + \lg(x+3)$. [4]

Thinking Process

To solve ✎ express the equation as a single term in lg, then remove the lg function.

Solution

$$\lg(7x-3) + 2\lg 5 = 2 + \lg(x+3)$$

$$\lg(7x-3) + \lg 5^2 - \lg(x+3) = 2$$

$$\lg\left(\frac{(7x-3)(25)}{(x+3)}\right) = 2$$

$$\lg\left(\frac{25(7x-3)}{(x+3)}\right) = 2\lg 10$$

$$\lg\left(\frac{25(7x-3)}{(x+3)}\right) = \lg 10^2$$

$$\frac{25(7x-3)}{(x+3)} = 100$$

$$7x - 3 = 4(x+3)$$

$$7x - 3 = 4x + 12$$

$$3x = 15$$

$x = 5$ **Ans.**

11 *(J10/P2/Q10)*

(a) Given that $\log_p X = 6$ and $\log_p Y = 4$, find the value of

(i) $\log_p\left(\dfrac{X^2}{Y}\right)$, [2]

(ii) $\log_Y X$. [2]

(b) Find the value of 2^z, where $z = 5 + \log_2 3$. [3]

(c) Express $\sqrt{512}$ as a power of 4. [2]

Thinking Process

(a) (i) Expand using $\log_a \frac{m}{n} = \log_a m - \log_a n$.

(ii) Express the given log in the base of p.

Apply $\log_b a = \dfrac{\log_c a}{\log_c b}$.

(b) To find 2^z ✎ find $\log_2 3$ by changing log to the base 10 and solve.

(c) Apply $(a^m)^n = a^{m \times n}$.

Solution

(a) (i) $\log_p\left(\dfrac{X^2}{Y}\right) = \log_p X^2 - \log_p Y$

$= 2\log_p X - \log_p Y$

$= 2(6) - 4 = 8$ **(Ans).**

(ii) $\log_Y X = \log_p X \div \log_p Y$

$= 6 \div 4 = 1.5$ **(Ans).**

(b) $z = 5 + \log_2 3$

$= 5 + \dfrac{\log_3}{\log_2} = 6.58496$

$\Rightarrow \quad 2^z = 2^{6.58496} = 96$

hence $2^z = 96$ **(Ans).**

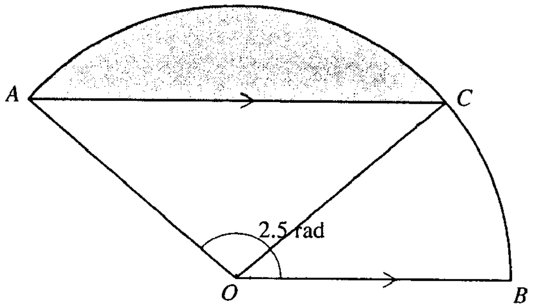
(c) $\sqrt{512} = (2^9)^{\frac{1}{2}}$

$= 2^{\frac{9}{2}} = \left((2^2)^{\frac{1}{2}}\right)^{\frac{9}{2}} = 4^{\frac{9}{4}} = 4^{2.25}$ **(Ans).**

Topic 9

Circular Measure

1 *(J06/P2/Q11)*



The diagram shows a sector *OACB* of a circle, centre *O*, in which angle $AOB = 2.5$ radians. The line *AC* is parallel to *OB*.

(i) Show that angle $AOC = (5 - \pi)$ radians. [3]

Given that the radius of the circle is 12 cm, find

(ii) the area of the shaded region, [3]

(iii) the perimeter of the shaded region. [3]

Thinking Process

(i) $\angle OAC = \pi - \angle AOB$. $\angle OAC = \angle ACO$ (base $\angle s$ of isosceles triangle).

(ii) Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$.

(iii) Find arc *AC* and length *AC* using cosine rule.

Solution

(i) $\angle OAC = (\pi - 2.5)$ rad (interior $\angle s$ between // lines)

$\angle OAC = \angle ACO$ ($OA = OC$, radii of circle)

$\therefore \angle AOC = \pi - 2\times(\pi - 2.5)$ ($\angle$ sum of Δ)
$= (\pi - 2\pi + 5)$ rad
$= (5 - \pi)$ rad (shown)

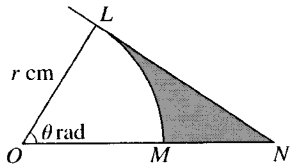
(ii) Area of shaded region

$= \frac{1}{2}(12)^2[(5-\pi) - \sin(5-\pi)]$
$= 72 \times (0.8995)$
$\approx 64.8 \text{ cm}^2$ (3sf)

(iii) arc $AC = 12 \times (5 - \pi)$ cm
≈ 22.30 cm (4sf)

$AC = \sqrt{12^2 + 12^2 - 2(12)(12)\cos(5-\pi)}$
$= 19.2$ cm (3sf)

2 *(D06/P2/Q12 Either)*



The diagram shows a sector of a circle, centre *O* and radius *r* cm. Angle *LOM* is θ radians. The tangent to the circle at *L* meets the line through *O* and *M* at *N*. The shaded region shown has perimeter *P* cm and area *A* cm². Obtain an expression, in terms of *r* and θ, for

(i) *P*, [4]

(ii) *A*. [3]

Given that $\theta = 1.2$ and that $P = 83$, find the value of

(iii) *r*, [2]

(iv) *A*. [1]

Thinking Process

(i) Find arc *LM*, *LN* and *MN*.

(ii) Find area of sector *LOM* and area of Δ.

(iii) Substitute the value of *P* and θ into the expression of *P*.

(iv) Substitute the value of *r* found in (iii) and the value of θ into the expression of *A*.

Solution

(i) Arc $LM = r \cdot \theta$ cm

In ΔOLN, $\angle OLN$ is 90° (tangent $\perp$ radius)

$\Rightarrow \cos\theta = \frac{r}{ON}$

$ON = \frac{r}{\cos\theta}$

$MN = ON - OM$

$= \frac{r}{\cos\theta} - r$

$\tan\theta = \frac{LN}{r}$

$LN = r\tan\theta$

$\therefore P = \text{arc } LM + MN + LN$

$= r\theta + \frac{r}{\cos\theta} - r + r\tan\theta$

$= r\left(\theta + \frac{1}{\cos\theta} - 1 + \tan\theta\right)$

(ii) Area of sector $LOM = \frac{1}{2}r^2\theta \text{ cm}^2$

Area of $\Delta LOM = \frac{1}{2} \times LN \times OL$

$= \frac{1}{2} \times r\tan\theta \times r = \frac{r^2}{2}\tan\theta$

Area of shaded region $= \frac{r^2}{2}\tan\theta - \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2(\tan\theta - \theta)\ \text{cm}^2$

(iii) $P = 83, \quad \theta = 1.2$

$\Rightarrow 83 = r\left(1.2 + \frac{1}{\cos 1.2} - 1 + \tan 1.2\right)$

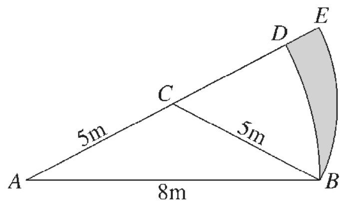
$\Rightarrow r = \dfrac{83}{0.2 + \frac{1}{\cos 1.2} + \tan 1.2}$

≈ 15.0 cm (3sf)

(iv) $A = \frac{1}{2}(15.0)^2[\tan 1.2 - 1.2]\ \text{cm}^2$

$\approx 154\ \text{cm}^2$ (3sf)

3 *(J07/P1/Q10)*



The diagram shows an isosceles triangle ABC in which $AB = 8$ m, $BC = CA = 5$ m. $ABDA$ is a sector of the circle, centre A and radius 8m. $CBEC$ is a sector of the circle, centre C and radius 5m.

(i) Show that angle BCE is 1.287 radians correct to 3 decimal places. [2]
(ii) Find the perimeter of the shaded region. [4]
(iii) Find the area of the shaded region. [4]

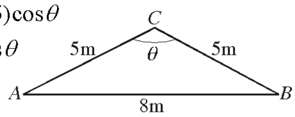
Thinking Process

(i) To find *BCE* ✎ Consider triangle *ACB* and find angle *ACB* by applying cosine rule.
(ii) ✎ Find arc length *BD* with *A* as centre and arc length *BE* with *C* as centre.

Arc length: $S = r\theta$. Area of a sector $= \frac{1}{2}r^2\theta$.

Solution

(i) Applying cosine rule on ΔABC



$8^2 = 5^2 + 5^2 - 2(5)(5)\cos\theta$

$64 = 25 + 25 - 50\cos\theta$

$14 = -50\cos\theta$

$\cos\theta = -\frac{14}{50} \quad \Rightarrow \quad \theta = 1.855$

$\therefore \quad \angle ACB = 1.855$ radians

now, $\angle ACB + \angle BCE = \pi$

$\Rightarrow \quad \angle BCE = \pi - \angle ACB$

$= \pi - 1.855 = 1.287$ radians **Shown**

(ii) $\angle CAB + \angle CBA = \angle BCE$

$2\angle CAB = \angle BCE \quad (\because \angle CAB = \angle CBA)$

$\therefore \quad \angle CAB = \frac{1.287}{2} = 0.6435$

Also, $AC + CD = AD$

$\Rightarrow \quad 5 + CD = 8 \quad \Rightarrow \quad CD = 3$

$CD + DE = CE$

$\Rightarrow \quad 3 + DE = 5 \quad \Rightarrow \quad DE = 2$

Perimeter of the shaded region

$=$ arc length BD + arc length $BE + DE$

$= (8)(0.6435) + (5)(1.287) + 2$

$= 5.148 + 6.435 + 2 = 13.583 \approx 13.6$ m **Ans**

(iii) Area of region CBD

$=$ area of sector $ABD -$ area of ΔABC

$= \frac{1}{2}(8)^2(0.6435) - \frac{1}{2}(8)(5)\sin(0.6435)$

$= 20.592 - 11.999 = 8.593\ \text{m}^2$

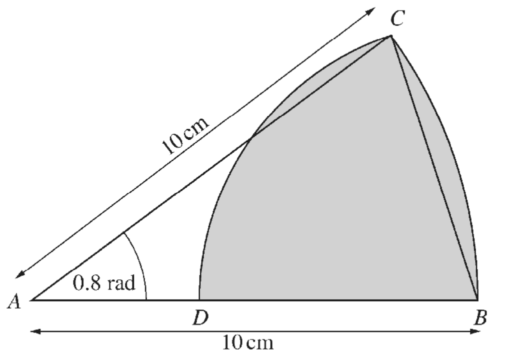
Area of shaded region

$=$ area of sector $CBE -$ area of region CBD

$= \frac{1}{2}(5)^2(1.287) - 8.593$

$= 16.08 - 8.593 = 7.487 \approx 7.49\ \text{m}^2$ **Ans**

4 *(N07/P1/Q10)*



The diagram shows a sector ABC of the circle, centre A and radius 10 cm, in which angle $BAC = 0.8$ radians. The arc CD of a circle has centre B and the point D lies on AB.

(i) Show that the length of the straight line BC is 7.79 cm, correct to 2 decimal places. [2]
(ii) Find the perimeter of the shaded region. [4]
(iii) Find the area of the shaded region. [4]

Thinking Process

(i) Apply cosine rule.
(ii) Perimeter of the shaded region = arc length *CD* + arc length *BC* + length of *BD*.
(iii) Area of shaded region = area of sector *BCD* + (area of sector *ABC* – area of triangle *ABC*)

Solution

(i) Applying cosine rule on ΔABC

$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos BAC$

$BC^2 = (10)^2 + (10)^2 - 2(10)(10)\cos(0.8)$

$BC^2 = 200 - 200(0.6967)$

$BC^2 = 60.66$

$BC = 7.79$ (to 2 dp) **Shown.**

(ii) ΔABC is an isosceles triangle

$\therefore \quad A\hat{B}C = A\hat{C}B$

$\Rightarrow \quad A\hat{B}C + A\hat{C}B + B\hat{A}C = \pi$

$2A\hat{B}C = \pi - 0.8 \quad \Rightarrow \quad A\hat{B}C = 1.17$ rad.

perimeter of the shaded region

= arc length DC + arc length $CB + BD$

$= (BC)(A\hat{B}C) + (AB)(B\hat{A}C) + BD$

$= (7.79)(1.17) + (10)(0.8) + 7.79 \qquad (\because BD = BC)$

$= 9.1143 + 8 + 7.79 = 24.9$ cm **Ans.**

(iii) Area of the shaded region

= area of sector ABC − area of ΔABC + area of sector BCD

$= \frac{1}{2}(10)^2(0.8) - \frac{1}{2}(10)(10)\sin 0.8 + \frac{1}{2}(7.79)^2(1.17)$

$= 40 - 35.87 + 35.50$

$= 39.63 \approx 39.6$ (to 3 sf) **Ans.**

5 *(N07/P2/Q6)*

Two circular flower beds have a combined area of $\frac{29\pi}{2}$ m^2. The sum of the circumferences of the two flower beds is 10π m. Determine the radius of each flower bed. [6]

Thinking Process

Area of a circle with radius r is πr^2 and circumference is $2\pi r$.

Solution

Let r_1 and r_2 be the radius of circular bed 1 & 2, respectively.

Area of circular bed 1 $= A_1 = \pi r_1^2$

Area of circular bed 2 $= A_2 = \pi r_2^2$

Given that, $\pi r_1^2 + \pi r_2^2 = \frac{29}{2}\pi$

$\Rightarrow \qquad r_1^2 + r_2^2 = \frac{29}{2} \cdots\cdots \text{(i)}$

Similarly, circumference of circular bed 1 & 2, is $2\pi r_1$ and $2\pi r_2$ respectively.

$\therefore \quad 2\pi r_1 + 2\pi r_2 = 10\pi$

$\Rightarrow \qquad r_1 + r_2 = 5 \cdots\cdots \text{(ii)}$

from eq. (ii): $r_2 = 5 - r_1$

subst into eq. (i): $r_1^2 + (5 - r_1)^2 = \frac{29}{2}$

$r_1^2 + 25 - 10r_1 + r_1^2 = \frac{29}{2}$

$2r_1^2 + 25 - 10r_1 = \frac{29}{2}$

$4r_1^2 + 50 - 20r_1 - 29 = 0$

$4r_1^2 - 20r_1 + 21 = 0$

using quadratic formula

$r_1 = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(21)}}{2(4)}$

$= \frac{20 \pm \sqrt{400 - 336}}{8}$

$= \frac{20 \pm \sqrt{64}}{8} = \frac{20 \pm 8}{8}$

$\therefore \quad r_1 = \frac{28}{8} \quad$ or $\quad r_1 = \frac{12}{8}$

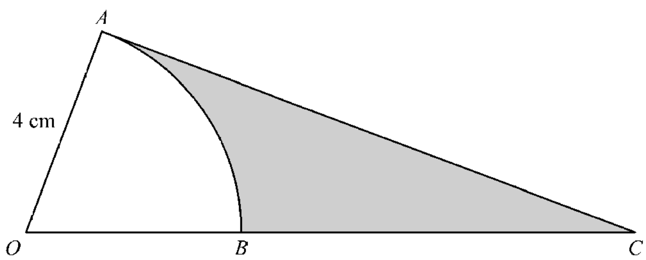
$\Rightarrow \quad r_1 = \frac{7}{2} = 3.5 \quad$ or $\quad r_1 = \frac{3}{2} = 1.5$

when $r_1 = 3.5$: $\quad r_2 = 5 - 3.5 = 1.5$

when $r_1 = 1.5$: $\quad r_2 = 5 - 1.5 = 3.5$

$\therefore$ radius of one bed is 1.5 m and other is 3.5 m. **Ans.**

6 *(J08/P1/Q7)*



The diagram shows a sector OAB of a circle, centre O, radius 4 cm. The tangent to the circle at A meets the line OB extended at C. Given that the area of the sector OAB is 10 cm², calculate

(i) the angle AOB in radians, [2]

(ii) the perimeter of the shaded region. [4]

Thinking Process

(i) Apply area of sector $= \frac{1}{2}r^2\theta$

(ii) Use $S = r\theta$ to find arc length AB. Line AC is tangent to the circle at point A. Therefore use right angled triangle OAC to find AC and OC.

Solution

(i) Area of sector $OAB = \frac{1}{2}r^2\theta$

$\Rightarrow\ 10 = \frac{1}{2}(4)^2\theta$

$10 = 8\theta$

$\theta = \frac{10}{8} = \frac{5}{4} = 1.25$ radian **Ans.**

(ii) Arc length $AB = r\theta = (4)\left(\frac{5}{4}\right) = 5$ cm

In ΔOAC, $\tan A\hat{O}C = \frac{AC}{OA}$

$\Rightarrow\ AC = (OA)\tan A\hat{O}C$

$\Rightarrow\ AC = (4)\tan\left(\frac{5}{4}\right)$

$\Rightarrow\ AC = 12.04$ cm

$OC^2 = OA^2 + AC^2$

$OC = \sqrt{(OA)^2 + (AC)^2}$

$= \sqrt{(4)^2 + (12.04)^2} = \sqrt{160.962} = 12.687$

$BC = OC - OB$

$= 12.687 - 4 = 8.687$ cm

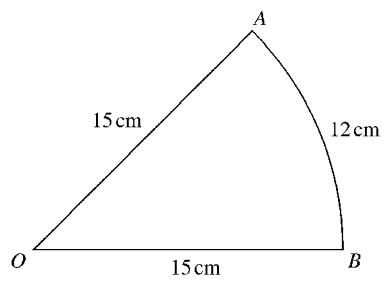
Perimeter of the shaded region

$=$ arc length $AB + BC + AC$

$= 5 + 8.687 + 12.04$

$= 25.727 \approx 25.7$ cm **Ans.**

7 *(J09/P1/Q1)*



The diagram shows a sector AOB of a circle, centre O, radius 15 cm. The length of the arc AB is 12 cm.

(i) Find, in radians, angle AOB. [2]

(ii) Find the area of the sector AOB. [2]

Thinking Process

(i) Apply formula, $S = r\theta$.

(ii) Apply formula, $A = \frac{1}{2}r^2\theta$.

Solution

(i) Arc length $= r\theta$

$12 = 15\theta$

$\theta = \frac{12}{15} = 0.8$ rad. **Ans.**

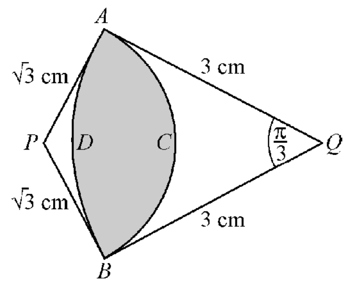
(ii) Area of sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}(15)^2(0.8) = 90\text{ cm}^2$ **Ans.**

8 *(N09/P2/Q11)*

Answer only **one** of the following two alternatives.

EITHER



In the diagram, ACB is an arc of a circle with centre P, and ADB is an arc of a circle with centre Q. Angle $AQB = \frac{\pi}{3}$, $AQ = BQ = 3$ cm and $AP = BP = \sqrt{3}$ cm.

(i) Show that angle $APB = \frac{2\pi}{3}$. [2]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [5]

Thinking Process

(i) Sum of angles of a quadrilateral = 2π radians. Note that AP and BP are perpendicular to AQ and BQ respectively.

(ii) To find perimeter of the shaded region find arc lengths of ADB and ACB using $s = r\theta$.

(iii) Find the area of two sectors $ACBP$ and $ADBQ$ and subtract the area of their respective triangles.

Solution

(i) AP and PB are tangents to the arc of circle with centre Q.

Therefore $AP \perp AQ$ and $BP \perp BQ$.

$\Rightarrow\ P\hat{A}Q = P\hat{B}Q = \frac{\pi}{2}$

In quadilateral $APBQ$

$A\hat{P}B + P\hat{B}Q + A\hat{Q}B + P\hat{A}Q = 2\pi$

$\Rightarrow\ A\hat{P}B + \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{2} = 2\pi$

$A\hat{P}B + \frac{4\pi}{3} = 2\pi$

$A\hat{P}B = 2\pi - \frac{4\pi}{3}$

$= \frac{2\pi}{3}$ radians **Shown.**

Topic 16

Integration and Applications

1 *(J06/P1/Q7)*

A particle moves in a straight line, so that, t s after leaving a fixed point O, its velocity, v ms^{-1}, is given by

$$v = pt^2 + qt + 4,$$

where p and q are constants. When $t = 1$ the acceleration of the particle is 8 ms^{-2}. When $t = 2$ the displacement of the particle from O is 22 m. Find the value of p and of q. [7]

Thinking Process

Differentiate v to get acceleration formula.
Integrate v to get displacement formula.
Substitute given values to solve for p and q.

Solution

$v = pt^2 + qt + 4$
$a = 2pt + q$
At $t = 1$, $a = 8$
$\Rightarrow 2p(1) + q(1) = 8$
$\Rightarrow 2p + q = 8$
$\Rightarrow \quad q = 8 - 2p$.....................(1)

$s = \int (pt^2 + qt + 4)\,dt$
$s = \frac{pt^3}{3} + \frac{qt^2}{2} + 4t + c$
At $t = 0$, $s = 0$,
$\Rightarrow c = 0$
$\therefore\ s = \frac{p}{3}t^3 + \frac{q}{2}t^2 + 4t$

At $t = 2$, $s = 22$,
$\Rightarrow \frac{p}{3}(2)^3 + \frac{q}{2}(2)^2 + 4(2) = 22$
$\Rightarrow \frac{8p}{3} + 2q + 8 = 22$
$\Rightarrow 8p + 6q = 42$
$\Rightarrow 4p + 3q = 21$........................(2)

Sub. (1) into (2):
$4p + 3(8 - 2p) = 21$
$\Rightarrow 4p + 24 - 6p = 21$
$\Rightarrow -2p = -3$
$p = \frac{3}{2}$
$p = 1\frac{1}{2}$

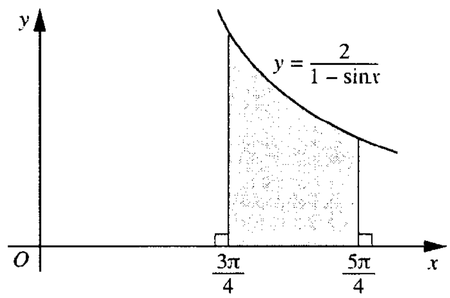
Sub. $p = 1\frac{1}{2}$ in (1):
$q = 8 - 2\left(1\frac{1}{2}\right)$
$= 8 - 3$
$= 5$

2 *(J06/P1/Q8)*

(i) Given that $y = \frac{1 + \sin x}{\cos x}$,

show that $\frac{dy}{dx} = \frac{1}{1 - \sin x}$. [5]

(ii)



The diagram shows part of the curve $y = \frac{2}{1 - \sin x}$.
Using the result given in part (i), find the area of the shaded region bounded by the curve, the x-axis and the lines $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$. [3]

Thinking Process

(i) Apply quotient rule to differentiate y.
(ii) Integrate the curve from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$. Apply reverse-differentiation to integrate y.

Solution with TEACHER's COMMENTs

(i) $y = \dfrac{1+\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\cos x(\cos x) - (1+\sin x)(-\sin x)}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1+\sin x}{1-\sin^2 x}$$

$$= \frac{1+\sin x}{(1+\sin x)\cdot(1-\sin x)}$$

$$= \frac{1}{1-\sin x} \quad \text{(shown)}$$

(ii) Area of shaded region

$$= \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{2}{1-\sin x}\, dx$$

$$= 2\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1-\sin x}\, dx$$

$$= 2\left[\frac{1+\sin x}{\cos x}\right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}$$

$$= 2\left[\left(\frac{1+\sin\frac{5\pi}{4}}{\cos\frac{5\pi}{4}}\right) - \left(\frac{1+\sin\frac{3\pi}{4}}{\cos\frac{3\pi}{4}}\right)\right]$$

$$= 2\left[\frac{1-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} - \frac{1+\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right]$$

$$= 2\left[\frac{-\frac{2}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right]$$

$$= 2\left(\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{1}\right)$$

$$= 2 \times 2$$

$$= 4 \text{ units}^2$$

Note that calculator must be in radian mode.

3

(J06/P2/Q4b)

(b) Evaluate $\int_0^{\frac{1}{2}} e^{1-2x}\, dx$. [4]

Thinking Process

(b) $\int_a^b e^{f(x)}\, dx = \left[\dfrac{e^{f(x)}}{f'(x)}\right]_a^b$.

Solution

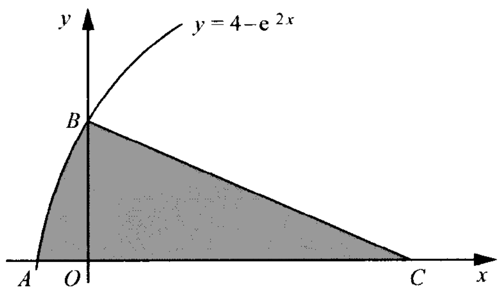
(b) $\int_0^{\frac{1}{2}} e^{1-2x}\, dx = -\frac{1}{2}\left[\left(e^{1-2x}\right)\right]_0^{\frac{1}{2}}$

$$= -\frac{1}{2}\left[e^0 - e^1\right]$$

$$\approx 0.859$$

4

(D06/P1/Q12 Either)



The diagram shows part of the curve $y = 4 - e^{-2x}$ which crosses the axes at A and at B.

(i) Find the coordinates of A and of B. [2]

The normal to the curve at B meets the x-axis at C.

(ii) Find the coordinates of C. [4]

(iii) Show that the area of the shaded region is approximately 10.3 square units. [5]

Thinking Process

(i) For A, solve $y = 0$. For B, solve for y when $x = 0$.
(ii) Find equation of normal at B. This line cuts the x-axis at C. Find C.
(iii) Integrate the curve from A to the origin and add that to the area of triangle formed by BC, x-axis and y-axis.

Solution

(i) $x = 0,\quad y = 4 - e^0$
$= 3$

$\therefore$ B is $(0, 3)$

$y = 0,\quad \Rightarrow 4 - e^{-2x} = 0$

$\Rightarrow e^{-2x} = 4$

$\Rightarrow -2x = \ln 4$

$\therefore\ x = \dfrac{\ln 4}{-2}$

$= \ln 4^{-\frac{1}{2}} = \ln\dfrac{1}{2}$

$\therefore$ A is $\left(\ln\dfrac{1}{2}, 0\right)$

(ii) $y = 4 - e^{-2x}$

$\frac{dy}{dx} = 2e^{-2x}$

At B (0, 3), gradient of tangent $= 2$

$\Rightarrow$ gradient of normal $= -\frac{1}{2}$

Equation of normal: $y = -\frac{1}{2}x + c$

$3 = c$

$\therefore\ y = -\frac{1}{2}x + 3$

At $y = 0$, $\Rightarrow -\frac{1}{2}x + 3 = 0$

$\Rightarrow \frac{1}{2}x = 3$

$\Rightarrow x = 6$

$\therefore\ C$ is $(6,\ 0)$

(iii) Area of shaded region

$= \int_{\ln\frac{1}{2}}^{0} (4 - e^{-2x})\ dx + \frac{1}{2}(6)(3)$

$= \left[4x - \frac{e^{-2x}}{(-2)}\right]_{\ln\frac{1}{2}}^{0} + 9$

$= \left[4x + \frac{1}{2e^{2x}}\right]_{\ln\frac{1}{2}}^{0} + 9$

$= \left(0 + \frac{1}{2}\right) - \left(4\ \ln\frac{1}{2} + \frac{1}{2e^{2\ln\frac{1}{2}}}\right) + 9$

$= \frac{1}{2} - 4\ \ln\frac{1}{2} - 2 + 9$

≈ 10.3 units^2 (shown)

5 *(D06/P2/Q3)*

Evaluate $\int_0^{\frac{\pi}{6}} \sin\left(2x + \frac{\pi}{6}\right) dx$. [4]

Thinking Process

Note: $\int a \sin bx\, dx = -\frac{a}{b}\cos bx + c$.

Solution

$\int_0^{\frac{\pi}{6}} \sin\left(2x + \frac{\pi}{6}\right) dx = -\frac{1}{2}\cdot\left[\cos\left(2x + \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}}$

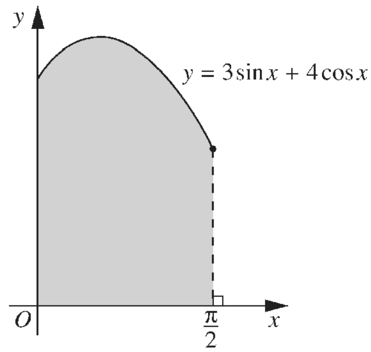
$= -\frac{1}{2}\left[\cos\left(\frac{2\pi}{6} + \frac{\pi}{6}\right) - \cos\frac{\pi}{6}\right]$

$= -\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{6}\right)$

$= -\frac{1}{2}\left(0 - \frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4}$

6 *(J07/P1/Q11(ii) Either)*



The graph shows part of the curve

$y = 3\sin x + 4\cos x$ for $0 \le x \le \frac{\pi}{2}$ radians.

Find the area of the shaded region. [5]

Thinking Process

To find area of the shaded region ✐ integrate the equation of the curve from $x = 0$ to $x = \frac{\pi}{2}$

Solution

Area of shaded region

$= \int_0^{\frac{\pi}{2}} (3\sin x + 4\cos x)\ dx$

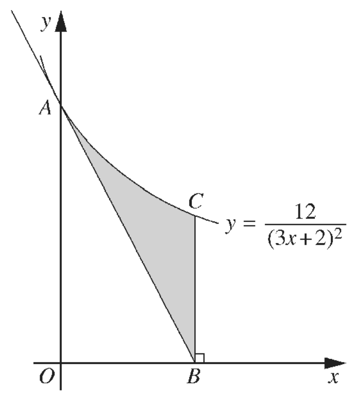
$= \left[-3\cos x + 4\sin x\right]_0^{\frac{\pi}{2}}$

$= (-3\cos(\frac{\pi}{2}) + 4\sin(\frac{\pi}{2})) - (-3\cos(0) + 4\sin(0))$

$= (0 + 4) - (-3 + 0)$

$= 4 + 3 = 7$ unit^2 **Ans**

7 *(J07/P1/Q11 Or)*



The diagram, which is not drawn to scale, shows part of the curve $y = \frac{12}{(3x+2)^2}$, intersecting the y-axis at A. The tangent to the curve at A meets the x-axis at B.

The point C lies on the curve and BC is parallel to the y-axis.

(i) Find the x-coordinate of B. [4]

(ii) Find the area of the shaded region. [6]

Thinking Process

(i) To find coordinates of point B ✐ Find the equation of the tangent using gradient and point A. The tangent meets the x-axis at B, so put $y = 0$ in the equation of the tangent.

(ii) To find the shaded area h Integrate the equation of the curve and subtract area of triangle AOB from it.

Solution

(i) $y = \dfrac{12}{(3x+2)^2}$. At A, $x = 0$

$\Rightarrow\ y = \dfrac{12}{(3(0)+2)^2} = \dfrac{12}{4} = 3$

$\therefore$ coordinates of point A are (0, 3)

Differentiating equation of the curve w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(12(3x+2)^{-2}\right)$$

$$= -24(3x+2)^{-3}(3) = -\frac{72}{(3x+2)^3}$$

at $x = 0$, $\dfrac{dy}{dx} = -\dfrac{72}{(3(0)+2)^3} = -\dfrac{72}{8} = -9$

$\therefore$ gradient of the tangent at point A is -9

equation of the tangent:

$y - 3 = -9(x - 0)$

$y - 3 = -9x \Rightarrow y + 9x = 3$

the tangent meets x-axis at B

$\therefore$ at B, $y = 0$

$\Rightarrow\ 0 + 9x = 3 \Rightarrow x = \dfrac{1}{3}$ **Ans**

(ii) BC is parallel to y-axis, therefore x-coordinates of B and C are same i.e. $\dfrac{1}{3}$

Area of shaded region

= area under the curve − area of ΔAOB

$$= \int_0^{\frac{1}{3}} \frac{12}{(3x+2)^2}\,dx - \frac{1}{2}\times\frac{1}{3}\times 3$$

$$= 12\int_0^{\frac{1}{3}} (3x+2)^{-2}\,dx - \frac{1}{2}$$

$$= 12\left[\frac{(3x+2)^{-1}}{(-1)(3)}\right]_0^{\frac{1}{3}} - \frac{1}{2}$$

$$= -4\left[\frac{1}{(3x+2)}\right]_0^{\frac{1}{3}} - \frac{1}{2}$$

$$= -4\left[\frac{1}{(3(\frac{1}{3})+2)} - \frac{1}{(3(0)+2)}\right] - \frac{1}{2}$$

$$= -4\left[\frac{1}{3} - \frac{1}{2}\right] - \frac{1}{2}$$

$$= -4\left[-\frac{1}{6}\right] - \frac{1}{2} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}\ \text{unit}^2$$ **Ans**

8 *(J07/P2/Q11)*

A particle, moving in a straight line, passes through a fixed point O with velocity 14 ms^{-1}. The acceleration, a ms^{-2}, of the particle, t seconds after passing through O, is given by $a = 2t - 9$. The particle subsequently comes to instantaneous rest, firstly at A and later at B. Find

(i) the acceleration of the particle at A and at B, [4]

(ii) the greatest speed of the particle as it travels from A to B, [2]

(iii) the distance AB. [4]

Thinking Process

(i) To find the acceleration ✐ find the equation for the velocity of the particle. Calculate the time when the velocity of the particle is zero.

(ii) Note that at greatest speed, acceleration of the particle is zero.

(iii) To find the distance AB ✐ first calculate the equation for the displacement.

Solution

(i) $a = 2t - 9$

velocity, $v = \int a\,dt$

$= \int (2t - 9)\,dt$

$= \dfrac{2t^2}{2} - 9t + C$

$= t^2 - 9t + C$

when $t = 0$, $v = 14$ m/s

$\Rightarrow\ 14 = (0)^2 - 9(0) + C \Rightarrow C = 14$

$\therefore\ v = t^2 - 9t + 14$

when a particle comes to rest instantaneously,

$v = 0$

$\Rightarrow\ t^2 - 9t + 14 = 0$

$(t-7)(t-2) = 0$

$t = 7$ and $t = 2$

$\Rightarrow$ the particle is at A when $t = 2$ seconds and it is at B when $t = 7$ seconds.

$\therefore$ acc. at A: $a = 2(2) - 9 = -5$ m/s^2 **Ans.**

acc. at B: $a = 2(7) - 9 = 5$ m/s^2 **Ans.**

(ii) Given, $a = 2t - 9$

At greatest speed, $a = 0$

$\Rightarrow \quad 2t - 9 = 0 \quad \Rightarrow \quad t = \frac{9}{2}$ seconds

now, $v = t^2 - 9t + 14$

at $t = \frac{9}{2}$

$v = (\frac{9}{2})^2 - 9(\frac{9}{2}) + 14$

$= \frac{81}{4} - \frac{81}{2} + 14 = -\frac{25}{4} = -6.25$ m/s **Ans.**

(iii) Let s be the distance

$\therefore \quad s = \int v\, dt$

$\Rightarrow \quad s = \int (t^2 - 9t + 14)\, dt$

$\Rightarrow \quad s = \frac{t^3}{3} - 9(\frac{t^2}{2}) + 14t + K$

when $t = 0, \quad s = 0$

$\Rightarrow \quad 0 = \frac{(0)^3}{3} - 9(\frac{(0)^2}{2}) + 14(0) + K \quad \Rightarrow \quad K = 0$

$\therefore \quad s = \frac{t^3}{3} - 9(\frac{t^2}{2}) + 14t$

particle is at A when $t = 2$

$\therefore$ displacement at A is:

$s = \frac{(2)^3}{3} - 9(\frac{(2)^2}{2}) + 14(2)$

$= \frac{8}{3} - 18 + 28 = \frac{38}{3}$ m.

particle is at B when $t = 7$

$\therefore$ displacement at B is:

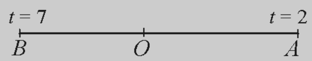
$s = \frac{(7)^3}{3} - 9(\frac{(7)^2}{2}) + 14(7)$

$= \frac{343}{3} - \frac{441}{2} + 98 = -\frac{49}{6}$ m.

$\therefore$ Distance $AB = \frac{38}{3} + \frac{49}{6}$

$= \frac{125}{6} = 20\frac{5}{6}$ m **Ans.**

Note that the negative sign in the displacement OB of the particle indicates that the particle covered $\frac{49}{6}$ metres in a direction opposite to that of OA.



Therefore total distance $AB = OA + OB$

9 *(N07/P2/Q9(ii))*

A particle travels in a straight line so that, t s after passing through a fixed point O, its speed, v ms^{-1}, is given by $v = 8\cos\left(\frac{t}{2}\right)$.

The particle first comes to instantaneous rest at the point P.

Find the distance OP. [4]

Thinking Process

To find the distance OP ✎ substitute $v = 0$ and solve for t. Integrate velocity to obtain an expression for displacement.

Solution

The particle comes to instantaneous rest at P

$\Rightarrow$ i.e. $v = 0$ at P

$\Rightarrow \quad 8\cos\left(\frac{t}{2}\right) = 0$

$\cos\frac{t}{2} = 0$

$\frac{t}{2} = \frac{\pi}{2}$

$\Rightarrow \quad t = \pi$ sec

$\therefore$ particle is at point P when $t = \pi$ sec

Distance $OP = \int_0^{\pi} v\, dt$

$= \int_0^{\pi} 8\cos\left(\frac{t}{2}\right) dt$

$= \left|8\sin\frac{t}{2} \times 2\right|_0^{\pi} = \left|16\sin\left(\frac{t}{2}\right)\right|_0^{\pi}$

$= 16\left(\sin\frac{\pi}{2} - \sin\frac{0}{2}\right)$

$= 16(1) = 16$ m (3 sf) **Ans.**

10 *(J08/P1/Q9)*

A curve is such that $\frac{dy}{dx} = 2\cos\left(2x - \frac{\pi}{2}\right)$. The curve passes through the point $\left(\frac{\pi}{2}, 3\right)$.

(i) Find the equation of the curve. [4]

(ii) Find the equation of the normal to the curve at the point where $x = \frac{3\pi}{4}$. [4]

Thinking Process

(i) To find the equation of curve ✎ integrate the given expression of gradient.

(ii) Find y at $x = \frac{3\pi}{4}$ from equation of curve obtain in part (i). Note that gradient of normal $= \frac{-1}{\text{grad. of tangent}}$. Apply $y - y_1 = m(x - x_1)$ to find equation of normal.

Solution

(i) $\dfrac{dy}{dx} = 2\cos\left(2x - \dfrac{\pi}{2}\right)$

$\Rightarrow \quad dy = 2\cos\left(2x - \dfrac{\pi}{2}\right) dx$

integrating both sides

$\int dy = \int 2\cos\left(2x - \dfrac{\pi}{2}\right) dx$

$y = 2\sin\left(2x - \dfrac{\pi}{2}\right) \times \dfrac{1}{2} + C$

$y = \sin\left(2x - \dfrac{\pi}{2}\right) + C$

the curve passes through $\left(\dfrac{\pi}{2},\ 3\right)$

$\therefore \quad 3 = \sin\left(2(\frac{\pi}{2}) - \dfrac{\pi}{2}\right) + C$

$3 = \sin\left(\dfrac{\pi}{2}\right) + C \qquad \because \sin\dfrac{\pi}{2} = 1$

$3 = 1 + C \quad \Rightarrow \quad C = 2$

$\therefore$ equation of curve is: $y = \sin\left(2x - \dfrac{\pi}{2}\right) + 2$

(ii) Equation of the curve from part (i) is:

$y = \sin\left(2x - \dfrac{\pi}{2}\right) + 2$

when $x = \dfrac{3\pi}{4}$,

$y = \sin\left(2(\frac{3\pi}{4}) - \dfrac{\pi}{2}\right) + 2$

$y = \sin\left(\dfrac{3\pi}{2} - \dfrac{\pi}{2}\right) + 2$

$y = \sin(\pi) + 2 \qquad \because \sin\pi = 0$

$y = 2$

the gradient of tangent is

$\dfrac{dy}{dx} = 2\cos\left(2x - \dfrac{\pi}{2}\right)$

at $x = \dfrac{3\pi}{4}$, $\dfrac{dy}{dx} = 2\cos\left(2(\frac{3\pi}{4}) - \dfrac{\pi}{2}\right)$

$= 2\cos\left(\dfrac{3\pi}{2} - \dfrac{\pi}{2}\right)$

$= 2\cos(\pi)$

$= 2(-1) = -2$

$\Rightarrow$ gradient of normal $= \dfrac{-1}{-2} = \dfrac{1}{2}$

$\therefore$ equation of normal with gradient $\dfrac{1}{2}$, passing through $\left(\dfrac{3\pi}{4},\ 2\right)$ is:

$y - 2 = \dfrac{1}{2}\left(x - \dfrac{3\pi}{4}\right) \quad \Rightarrow \quad y = \dfrac{1}{2}x - \dfrac{3}{8}\pi + 2$

11 *(J08/P2/Q10)*

(a) Find

(i) $\displaystyle\int \frac{12}{(2x-1)^4}\,dx$, [2]

(ii) $\displaystyle\int x(x-1)^2\,dx$. [3]

(b) (i) Given that $y = 2(x-5)\sqrt{x+4}$, show that $\dfrac{dy}{dx} = \dfrac{3(x+1)}{\sqrt{x+4}}$. [3]

(ii) Hence find $\displaystyle\int \frac{(x+1)}{\sqrt{x+4}}\,dx$. [2]

Thinking Process

(a) (i) Rewrite and integrate.
(ii) Expand and integrate.
(b) (i) Differentiate using product rule.
(ii) Apply anti-differentiation to find integral.

Solution

(a) (i) $\displaystyle\int \frac{12}{(2x-1)^4}\,dx = 12\int (2x-1)^{-4}\,dx$

$= 12\left(\dfrac{(2x-1)^{-3}}{-3(2)}\right) + C$

$= 12\left(\dfrac{1}{-6(2x-1)^3}\right) + C$

$= -\dfrac{2}{(2x-1)^3} + C$ **Ans.**

(ii) $\displaystyle\int x(x-1)^2\,dx = \int x(x^2 - 2x + 1)\,dx$

$= \displaystyle\int (x^3 - 2x^2 + x)\,dx$

$= \dfrac{x^4}{4} - 2\left(\dfrac{x^3}{3}\right) + \dfrac{x^2}{2} + C$

$= \dfrac{1}{4}x^4 - \dfrac{2}{3}x^3 + \dfrac{1}{2}x^2 + C$ **Ans.**

(b) (i) $y = 2(x-5)\sqrt{x+4}$

$= (2x-10)(x+4)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = (2x-10)\dfrac{d}{dx}(x+4)^{\frac{1}{2}} + (x+4)^{\frac{1}{2}}\dfrac{d}{dx}(2x-10)$

$= (2x-10)\left(\dfrac{1}{2}(x+4)^{-\frac{1}{2}}\right) + (x+4)^{\frac{1}{2}}(2)$

$= \dfrac{x-5}{\sqrt{x+4}} + 2\sqrt{x+4}$

$= \dfrac{(x-5) + 2(x+4)}{\sqrt{x+4}}$

$= \dfrac{x - 5 + 2x + 8}{\sqrt{x+4}}$

$= \dfrac{3x+3}{\sqrt{x+4}} = \dfrac{3(x+1)}{\sqrt{x+4}}$ **Proved.**

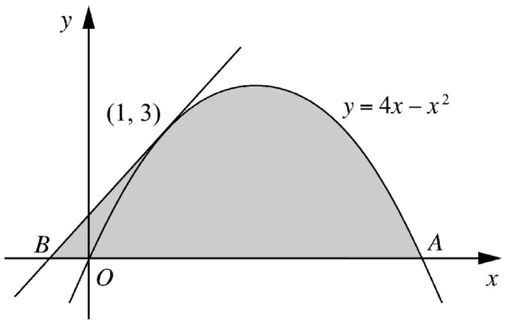
(ii) $\int \frac{(x+1)}{\sqrt{x+4}}\,dx = \frac{1}{3}\int \frac{3(x+1)}{\sqrt{x+4}}\,dx$

$= \frac{1}{3}\int \frac{d}{dx}\left(2(x-5)\sqrt{x+4}\right)dx$

$= \frac{1}{3}\left(2(x-5)\sqrt{x+4}\right)$

$= \frac{2}{3}(x-5)\sqrt{x+4}$ **Ans.**

12 *(J08/P2/Q12 Either)*



The diagram shows the curve $y = 4x - x^2$, which crosses the x-axis at the origin O and the point A.
The tangent to the curve at the point (1, 3) crosses the x-axis at the point B.

(i) Find the coordinates of A and of B. [5]

(ii) Find the area of the shaded region. [5]

Thinking Process

(i) To find coordinates of A ✐ put $y = 0$ in equation of curve. To find coordinates of B, find the equation of tangent, put $y = 0$ and find the value of x.

(ii) Total shaded area = area of triangle + area under the curve.

Solution

(i) $y = 4x - x^2$

at point A, $y = 0$

$\Rightarrow 4x - x^2 = 0$

$x(4-x) = 0$

$x = 0$ or $x = 4$

$\therefore$ coordinates of A (4, 0) **Ans.**

eq. of curve: $y = 4x - x^2$

$\frac{dy}{dx} = 4 - 2x$

at $x = 1$

$\frac{dy}{dx} = 4 - 2(1) = 2$

$\therefore$ gradient of tangent at point (1, 3) = 2

equation of tangent:

$y - 3 = 2(x-1)$

$y - 3 = 2x - 2$

$y = 2x + 1 \cdots\cdots\cdots$(i)

the tangent crosses x-axis at B

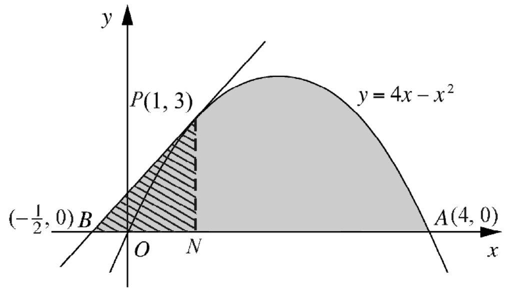
$\therefore$ at point B, $y = 0$, put in eq.(i)

$0 = 2x + 1$

$x = -\frac{1}{2}$

$\therefore$ coordinates of B $\left(-\frac{1}{2}, 0\right)$ **Ans.**

(ii)



Area of shaded region

= Area of ΔPBN + area under the curve

$= \frac{1}{2}(BN)(PN) + \int_1^4 (4x - x^2)\,dx$

$= \frac{1}{2}(1+\frac{1}{2})(3) + \left[4\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_1^4$

$= \frac{1}{2}(\frac{3}{2})(3) + \left[2x^2 - \frac{x^3}{3}\right]_1^4$

$= \frac{9}{4} + \left[\left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(1)^2 - \frac{(1)^3}{3}\right)\right]$

$= \frac{9}{4} + \left[\left(32 - \frac{64}{3}\right) - \left(2 - \frac{1}{3}\right)\right]$

$= \frac{9}{4} + \left[\frac{32}{3} - \frac{5}{3}\right]$

$= \frac{9}{4} + 9$

$= \frac{45}{4} = 11.25$ units^2 **Ans.**